PROBLEM SET 9

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Problem 1. *Let $G_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$, and if $f \in \mathcal{S}'(\mathbb{R}^n)$, let $u(x,t) = f * G_t(x)$. Prove*

(1) u satisfies $(\partial_t - \Delta)u = 0$ on $\mathbb{R}^n \times (0,\infty)$ and $u(\cdot,t) \to f$ in $\mathcal{S}'$ as $t \to 0$.
(2) If f is a tempered function, then $u(x,t) \to f(x)$ a.e. as $t \to 0$.

Sketch of proof. (1) We notice $G_t(x) = \frac{1}{t^{\frac{n}{2}}}G(\frac{x}{t^{1/2}})$ where $G(x) = \frac{1}{(4\pi)^{n/2}}e^{-\frac{|x|^2}{4}}$ is the well-known Gaussian distribution, so G_t is an approximate identity. Hence $u = f * G_t \to f$ in S' as $t \to \infty$. To show u satisfies the equation, we notice $(\partial_t - \Delta)f * G_t = f * (\partial_t - \Delta)G_t$, therefore it suffices to show $(\partial_t - \Delta)G_t = 0$ which is a straightforward calculation.

(2) This follows from the observation that G_t is an approximate identity. □

Problem 2. *Let W_t be the inverse Fourier transform of $(2\pi|\xi|)^{-1}\sin(2\pi|\xi|t)$. Prove the following.*

(1) If $n = 1$, $W_t = \frac{1}{2}\chi(-t,t)$
(2) If $n = 3$, let σ_R denote surface measure on the sphere $|x| = R$. Then $\hat{\sigma}_R(\xi) = 2R|\xi|^{-1}\sin(2\pi R|\xi|)$, and hence $W_t = (4\pi t)^{-1}\sigma_t$.
(3) If $n = 2$, think of $\xi \in \mathbb{R}^2$ as an element of $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. Transform the integral

$$\frac{2R\sin(2\pi R|\xi|)}{|\xi|} = \int_{|x|=R} e^{-2\pi i x\cdot\xi}d\sigma_R(x)$$

as an integral over the disc $D_R = \{y : |y| \le R\}$ in $\mathbb{R}^2$. Conclude that for $n = 2$,

$$W_t(x) = (2\pi)^{-1}(t^2 - |x|^2)^{-\frac{1}{2}}\chi_{D_t}(x).$$

Proof. (1) The Fourier transform of $\frac{1}{2}\chi(-t,t)$ is

$$\frac{1}{2}\int_{-t}^{t} e^{-2\pi i\xi x}dx = -\frac{1}{4\pi i\xi}[e^{-2\pi i\xi x}]_{-t}^{t} = \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}.$$

(2) We note $\hat{\sigma}_R(\xi)$ is radial, due to the symmetry of sphere. To elaborate, for any $\rho \in SO(3)$ we have

$$\hat{\sigma}_R(\rho\xi) = \int e^{-2\pi i x\cdot\rho\xi}d\sigma_R(x) = \int e^{-2\pi i(\rho x)\cdot\xi}d\sigma_R(x)$$

But since σ_R is invariant under $SO(3)$, we have $d\sigma_R(x) = d\sigma_R(\rho x)$, hence $\hat{\sigma}_R(\rho\xi) = \hat{\sigma}_R(\xi)$ for all $\rho \in SO(3)$, *i.e.* $\hat{\sigma}_R$ is radial. This means $\hat{\sigma}_R(\xi) = f(|\xi|)$ for some f. There are then several ways to determine f, one is to realize $f(|\xi|) = \hat{\sigma}_R((|\xi|,0,0))$ and use sphere coordinates to explicitly calculate f. Another way is to realize $\hat{\sigma}_R$ has to satisfy a differential equation, and hence so does f, then solve for f. Any natural differential equation $\hat{\sigma}_R$ satisfies must be symmetric under $SO(3)$, and the most natural differential operator invariant under $SO(3)$ is the Laplacian. So let's try to compute $\Delta\hat{\sigma}_R$. We have $\partial_k\hat{\sigma}_R(\xi) = \int -2\pi i x_k e^{-2\pi i x\cdot\xi} d\sigma_R(x)$, and $\partial_k^2\hat{\sigma}_R(\xi) = \int -4\pi^2 x_k^2 e^{-2\pi i x\cdot\xi} d\sigma_R(x)$, therefore

$$\Delta\hat{\sigma}_R(\xi) = \int -4\pi^2|x|^2 e^{-2\pi i x\cdot\xi} d\sigma_R(x) = -4\pi^2R^2\hat{\sigma}_R(\xi).$$

Now the Laplacian under spherical coordinate and that $\hat{\sigma}_R(\xi) = f(|\xi|)$ is radial, we have $\Delta\hat{\sigma}_R(\xi) = \frac{1}{r}\partial_r^2(rf)$ where $r = |\xi|$. So f satisfies the ordinary differential equation

$$r\frac{d^2}{dr^2}f + 2\frac{d}{dr}f + 4\pi^2R^2rf = 0.$$

It is easy to verify $f(r) = 2Rr^{-1}\sin(2\pi Rr)$ satisfy the equation, and hence is the unique solution by standard ODE theory. This proves $\hat{\sigma}_R(\xi) = f(|\xi|) = 2R|\xi|^{-1}\sin(2\pi R|\xi|)$.

(3) We write $x = (y, z)$ for $y = (y_1, y_2) \in \mathbb{R}^2$ and $z \in \mathbb{R}$ where $z^2 = R^2 - |y|^2$. So $d\sigma_R(x) = \sqrt{(\partial z/\partial y_1)^2 + (\partial z/\partial y_2)^2 + 1}dy = \frac{R}{\sqrt{R^2-|y|^2}}dy$. Therefore

$$\int_{|x|=R} e^{-2\pi i x\cdot\xi} d\sigma_R(x) = 2\int_{|y|\le R} e^{-2\pi i y\cdot\xi}\frac{R}{\sqrt{R^2-|y|^2}}dy = \hat{h_R}$$

where $h_R(y) = \frac{2R}{\sqrt{R^2-|y|^2}}\chi_{D_R}$ and the factor 2 reflects the symmetry of northern and southern hemisphere. So $\hat{W_R} = (2\pi|\xi|)^{-1}\sin(2\pi|\xi|R) = \frac{\pi}{R}\hat{h}_R$, therefore $W_R(y) = (2\pi)^{-1}(R^2 - |y|^2)^{-\frac{1}{2}}\chi_{D_R}(y)$.

□

Problem 3. *Solve $(\partial_t - \partial_x^2)u = 0$ on $(a, b) \times (0, \infty)$ with boundary conditions $u(x, 0) = f(x)$ on (a, b), $u(a, t) = u(b, t) = 0$ for $t > 0$. Solve this again, but with boundary condition $u(a, t) = u(b, t) = 0$ replaced by $\partial_x u(a, t) = \partial_x u(b, t) = 0$.*

Solution. We may assume $a = 0, b = \frac{1}{2}$. Since $f(0) = f(1/2) = 0$, we may extend f to an odd function with period 1 on $\mathbb{R}$. Then f has Fourier series $f(x) = \sum_{n=1}^{\infty} b_n \sin(2\pi nx)$. Assume solution $u(x, t)$ has the form $\sum\limits_{n=1}^{\infty} v_n(t)\sin(2\pi nx)$, then since $(\partial_t - \partial_x^2)u = 0$ we get $v_n'(t) = -4\pi^2n^2v_n(t)$ and $v_n(0) = b_n$ for all $n \in \mathbb{N}_+$ by comparing Fourier coefficients. Therefore $v_n(t) = b_ne^{-4\pi^2n^2t}$ and

$$u(x, t) = \sum_{n=1}^{\infty} b_ne^{-4\pi^2n^2t}\sin(2\pi nx).$$

Note that our logic here is in fact to first reasonably guess a potential solution, so it remains to verify the above one is indeed a solution. We need to check uniform convergence and ability to differentiate the series term by term. In the case f is smooth (C^2 or even weaker regularity is enough), this is not a problem, since the Fourier coefficients of u is the Fourier coefficients of f multiplied by a fast decreasing function $e^{-4\pi^2n^2t}$. Similarly in the second situation we extend f to an even function and write $f(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos(2\pi nx)$, then the solution is

$$u(x,t) = \frac{a_0}{2} + \sum_{n\geq 1} a_n e^{-4\pi^2 n^2 t} \cos(2\pi nx).$$

□

Problem 4. *If μ is a positive Borel measure on $\mathbb{T}^1$ with $\mu(\mathbb{T}^1) = 1$, show that $|\hat{\mu}(k)| < 1$ for all $k \neq 0$ unless μ is a convex combination of the point masses at $0, m^{-1}, ..., (m-1)m^{-1}$ for some $m \in \mathbb{N}$, in which case $\hat{\mu}(km) = 1$ for all $k \in \mathbb{Z}$.*

Proof. Let's first deal with the case where there exists $m \neq 0$ so that $\hat{\mu}(m) = 1$. Notice that for any measurable set A we have

$$\begin{aligned} 1 = \int_{\mathbb{T}^1} e^{-2\pi i m x} d\mu(x) &\leq \left|\int_A e^{-2\pi i m x} d\mu(x)\right| + \left|\int_{\mathbb{T}^1 - A} e^{-2\pi i m x} d\mu(x)\right| \\ &\leq \int_A |e^{-2\pi i m x}| d\mu(x) + \int_{\mathbb{T}^1 - A} |e^{-2\pi i m x}| d\mu(x) = \mu(A) + \mu(\mathbb{T}^1 - A) = 1 \end{aligned}$$

Hence $|\int_A e^{-2\pi i m x} d\mu(x)| = \mu(A)$ for all A. If $\mu(A) \neq 0$, then we may write $\int_A e^{-2\pi i m x} d\mu(x) = e^{i\theta_A}\mu(A)$ for some $\theta_A \in [0, 2\pi)$. Now suppose A, B are disjoint measurable with positive measures, then we claim $\theta_A = \theta_B$. Indeed, we have $e^{i\theta_A}\mu(A) + e^{i\theta_B}\mu(B) = e^{i\theta_{A\cup B}}\mu(A\cup B)$, therefore $|e^{i\theta_A}\mu(A) + e^{i\theta_B}\mu(B)| = |e^{i\theta_{A\cup B}}\mu(A\cup B)| = \mu(A) + \mu(B)$, or equivalently $|e^{i\theta_A - i\theta_B}\mu(A) + \mu(B)| = \mu(A) + \mu(B)$, it then follows $\theta_A = \theta_B$ since $\mu(A), \mu(B) > 0$. Moreover we see that $\theta_{A\cup B} = \theta_A = \theta_B$.

Next we show if $x_0 \in \operatorname{supp}\mu$, then $e^{-2\pi i m x_0} = 1$. Suppose $e^{-2\pi i m x_0} \neq 1$, then $\sin(2\pi i m x_0) \neq 0$, we may assume $\sin(2\pi m x) \geq 2\epsilon > 0$. Let U be an open neighborhood of x_0 on which $\sin(2\pi m x) \geq \epsilon > 0$. We would like to show $x_0 \notin \operatorname{supp}\mu$. Suppose for contradiction $x_0 \in \operatorname{supp}\mu$, we have $\mu(U) > 0$. We may shrink U so that $\mu(U) < 1$, otherwise $\operatorname{supp}\mu = \{x_0\}$ and thus $1 = \hat{\mu}(m) = e^{-2\pi i m x_0}$ which is impossible. So now both U and $\mathbb{T}^1 - U$ have positive measure, by previous discussion, we see $\theta_U = \theta_{\mathbb{T}^1 - U} = \theta_{\mathbb{T}^1} = 0$, *i.e.* $\int_U e^{-2\pi i m x} d\mu(x) = \mu(U)$. However the imaginary part of $\int_U e^{-2\pi i m x} d\mu(x)$ is $-\int_U \sin(2\pi m x) d\mu(x) \leq -\epsilon\mu(U) < 0$. Contradiction. This proves $\operatorname{supp}\mu \subset \{0, 1/m, \ldots, (m-1)/m\}$.

So μ must be positive linear combination of point masses at $0, 1/m, \ldots, (m-1)/m$, and the sum of coefficients is $\mu(\mathbb{T}^1) = 1$. In this case for $x \in \operatorname{supp}\mu$, $e^{-2\pi i m x} = 1 = e^{-2\pi i k m x}$, so we have $\hat{\mu}(km) = 1$ for all $k \in \mathbb{Z}$. This finishes the proof for the case $\hat{\mu}(m) = 1$.

In general, if $|\hat{\mu}(m)| = 1$, then $\hat{\mu}(m) = e^{2\pi i s}$. Let $\nu = e^{-2\pi i s}\mu$, then $\hat{\nu}(m) = 1$. Thus ν is a convex combination of point masses at $0, 1/m, \ldots, (m-1)/m$ and $\hat{\nu}(km) = 1$ for all $k \in \mathbb{Z}$. Therefore μ is a convex combination of point masses at $s, 1/m + s, \ldots (m-1)/m + s$ and $\mu(km) = e^{2\pi i s}$ for all $k \in \mathbb{Z}$.

□

Problem 5. *Show that if $\Delta(\mathbb{R}^n)$ is the set of finite linear combinations of point masses on $\mathbb{R}^n$, then $\Delta(\mathbb{R}^n)$ is vaguely dense in $M(\mathbb{R}^n)$.*

Proof. We first note $C_c(\mathbb{R}^n)$ is contained in the vague closure of $\Delta(\mathbb{R}^n)$, because for any $f \in C_c, g \in C_0$ we have $\int fg$ is approximated by Riemann sum, so f is the vague limit of linear combination of point masses at the centers of cubes that covers $\operatorname{supp}(f)$. Therefore, the vague closure of $C_c(\mathbb{R}^n)$ is contained in the vague closure of $\Delta(\mathbb{R}^n)$. It remains to show $C_c(\mathbb{R}^n)$ is dense in $M(\mathbb{R}^n)$. This can be proved in two steps. First, we notice $C_c(\mathbb{R}^n)$ is L^1-dense hence vaguely dense in $L^1(\mathbb{R}^n)$ thanks to Holder inequality. Second $L^1(\mathbb{R}^n)$ is vaguely dense in $M(\mathbb{R}^n)$, because for any $\mu \in M(\mathbb{R}^n)$, μ is the limit of $\mu * \phi_t \in L^1$ for approximate identity $\{\phi_t\}$ since for any $g \in C_0(\mathbb{R}^n)$

$$|\int g(x) \int \phi_t(x-y) d\mu(y) dm(x| \leq \|g\|_\infty \|\phi_t\|_1 \|\mu\| < \infty$$

and thus by Fubini theorem

$$\int g(x) \int \phi_t(x-y) d\mu(y) dm(x) = \int \int g(x) \phi_t(x-y) dm(x) d\mu(y) = \int g * \tilde{\phi}_t d\mu.$$

Now that $g \in C_0$ hence uniformly continuous, $g * \tilde{\phi}_t$ is uniformly convergent to g, so $\int g(\mu * \phi_t) dm \to \int g d\mu$ as $t \to 0$. $\square$